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Every 8-Uniform 8-Regular Hypergraph Is 2-Colorable

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Abstract. As is well known, Lovász Local Lemma implies that every d-uniform d-regular hypergraph is 2-colorable, provided $d \geq 9$. We present a different proof of a slightly stronger result; every d-uniform d-regular hypergraph is 2-colorable, provided $d \geq 8$.

1. Introduction

A d-uniform d-regular hypergraph is a hypergraph in which every edge contains precisely d vertices and every vertex is contained in precisely d edges. It is well known (see, e.g., [4], [6]) that for $d \geq 9$ each such hypergraph is 2-colorable, i.e., there is a 2-coloring of its vertices with no monochromatic edges. To the best of our knowledge, the only known proof of this fact is the one that applies the Lovász Local Lemma to show that a random vertex coloring of the given hypergraph with 2 colors contains no monochromatic edges with positve (though very small) probability. Let D be the set of all positive integers d, such that every d-uniform d-regular hypergraph is 2-colorable. One can easily check that if $d \in D$ then $d' \in D$ for every $d' \geq d$. Indeed, if $H' = (V, E')$ is a d'-uniform d'-regular hypergraph, then by Hall's theorem (cf. e.g., [5]), every edge $e' \in E'$ contains an edge $e(e') \subseteq e'$ of cardinality d such that the hypergraph $H = (V, \{e(e') : e' \in E'\})$ is d-uniform and d-regular. The 2-colorability of H implies that of H'.

Put $\delta = \min\{d : d \in D\}$. Then

$$4 \leq \delta \leq 9. \tag{1.1}$$

The upper bound follows from the result, stated above, that every 9-uniform 9-regular hypergraph is 2 colorable. The lower bound is a consequence of the fact that the hypergraph whose vertices are the seven points of the Fano plane and whose edges are the seven lines of this plane is 3-uniform, 3-regular and not 2-colorable.

In this note we slightly improve inequality (1.1) by proving the following theorem which gives $\delta \leq 8$;

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Theorem 1.1. *For every $d \geq 8$, every d-uniform d-regular hypergraph is* 2 *colorable.*

The most interesting feature of this result is not its statement, which is only a slight improvement of (1.1), but its proof, which is completely different from the probabilistic one that gives $\delta \leq 9$. It seems plausible that in fact every 4-uniform 4-regular hypergraph is 2-colorable, i.e., $\delta = 4$, but at the moment we are unable to improve the estimate $\delta \leq 8$ given by Theorem 1.1.

2. Proofs

A $(1,2)$*-factor* of a graph $G = (V, E)$ is a spanning subgraph F of G, every connected component of which is either a cycle or an isolated edge. Let us call such an F *even* if the number of its connected components is even, and *odd* otherwise.

The following lemma seems interesting in its own right. Its proof is very similar to the proof of the main result of Friedland in [3].

Lemma 2.1. *For every $d \geq 8$, every d-regular simple graph contains an even* $(1,2)$*-factor and an odd* $(1,2)$*-factor.*

Proof. Let $G = (V, E)$ be a d regular simple graph on a set V of n vertices. Let $A = (a_{uv})_{u,v \in V}$ be the adjacency matrix of G given by $a_{uv} = 1$ if $uv \in E$ and $a_{uv} = 0$ otherwise. One can easily check that the permanent, perm(A), of A is just the total number of $(1,2)$-factors of G. Similarly, the determinant, det(A), of A is precisely $(-1)^n$ times the difference between the number of even $(1,2)$-factors of G and that of odd $(1,2)$-factors of G. By the well known Van-der-Waerden conjecture, proved by Falikman and Egorichev ([2], [1]),

$$\text{perm}(A) \geq d^n \cdot \frac{n!}{n^n} \geq \left(\frac{d}{e}\right)^n.$$

By the classical Hadamard Inequality

$$|\det A| \leq d^{n/2}.$$

As $d \geq 8 > e^2$, we conclude that perm$(A) > |\det A|$ and hence both the number of even $(1,2)$-factors of G and that of odd $(1,2)$-factors of G are positive. This completes the proof of the lemma. □

Corollary 2.2. *For every $d \geq 8$, every d-regular simple bipartite graph G contains a* $(1,2)$*-factor, with one component being a cycle of length* $0 \pmod 4$, *and all the other components being isolated edges.*

Proof. Clearly G has an even number $2m$ of vertices. By Lemma 2.1, G contains an even $(1,2)$-factor and an odd $(1,2)$-factor. Let F be a $(1,2)$-factor of G whose number of connected components is not congruent to m modulo 2. As G is bipartite, each connected component of F has an even number of vertices. Let f_0 be the number of these components whose size is $0 \pmod 4$, and let f_2 be the number of these components whose size is $2 \pmod 4$. Clearly $2f_2 \equiv 2m \pmod 4$, i.e., $f_2 \equiv m \pmod 2$. By the choice of F this implies that f_0 is odd, and in particular F contains a component C of size $0 \pmod 4$. By omitting every second edge from each component of F other than C that has more than one edge, we get a $(1, 2)$-factor of G with the desired properties. □

We can now prove Theorem 1.1. Suppose $d \geq 8$, and let $H = (V(H), E(H))$ be a d-uniform d-regular hypergraph. Clearly we may assume that H is connected. Let G be the bipartite graph on the classes of vertices $V(H)$ and $E(H)$, in which $v \in V(H)$ is adjacent to $e \in E(H)$ iff, in H, e contains v. Clearly G is simple, connected and d-regular. By Corollary 2.2, G contains a $(1, 2)$-factor F, whose connected components are $C_0, C_1, C_2, \ldots, C_r$ where C_0 is a cycle containing an even number of members of $V(H)$ and an equal even number of members of $E(H)$, and each other C_i is an edge in G joining some $v_i \in V(H)$ to some $e_i \in E(H)$. As the number of members of $V(H)$ in C_0 is even, we can color them alternately by red and blue, such that each member of $E(H)$ in C_0 will have a red and a blue neighbor in C_0. Notice that this partial coloring of $V(H)$ satisfies the following condition:

> If for some i, $1 \leq i \leq r$, v_i is already colored, then e_i has in G both a red neighbor and a blue neighbor. (2.1)

Indeed, condition (2.1) is satisfied trivially, as no v_i is colored. Let us extend the given coloring to some maximal coloring l satisfying (2.1), i.e., for any additional coloration of some uncolored v_i, condition (2.1) will be violated. We claim that l is a coloring of the whole set $V(H)$. To see this assume that it is not, and that some v_i's $(1 \leq i \leq r)$ are not colored. Let R be the induced subgraph of G on $\bigcup \{C_i\colon v_i$ is not colored$\}$. As G is connected, there are edges of G that join vertices of R to vertices not in R. However, since G is regular and bipartite, this implies that there is some i with $C_i \subset R$, such that there is an edge from e_i to some vertex not in R, i.e., to some member $v \in V(H)$ which is already colored under the partial coloring l. We can now color v_i blue if v is colored red and red if v is colored blue to obtain a new partial coloring l' satisfying condition (2.1). This contradicts the maximality of l and establishes the claim.

Returning to the hypergraph H, we now color each of its vertices $v \in V(H)$ by the color it received under l. As in G each vertex corresponding to an edge $e \in E(H)$ has both a red neighbor and a blue neighbor, the above coloring of the hypergraph H contains no monochromatic edges. This completes the proof of Theorem 1.1. □

References

1. Egorichev, G.P.: The solution of the Van der Waerden problem for permanents. Dokl. Acad. Nauk SSSR **258**, 1041–1044 (1981)
2. Falikman, D.I.: A proof of Van der Waerden's conjecture on the permanent of a doubly stochastic matrix. Mat. Zametki **29**, 931–938 (1981)
3. Friedland, S.: Every 7-regular digraph contains an even cycle. (to appear)
4. Graham, R.L., Rothschild, B.L., Spencer, J.H.: Ramsey Theory. p. 90. New York: Wiley Interscience 1980
5. Hall, M.: Combinatorial Theory. pp. 45–46. New York: Wiley 1967
6. Spencer, J.: Ten Lectures on the Probabilistic Method. p. 59. Philadelphia: SIAM 1987

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